

# Teleparallel Energy-Momentum Distribution of Locally Rotationally Symmetric Spacetimes

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## Abstract

In this paper, we explore the energy-momentum distribution of locally rotationally symmetric (LRS) spacetimes in the context of the teleparallel theory of gravity by considering the three metrics, I, II and III, representing the whole class of LRS spacetimes. In this regard, we use the teleparallel versions of the Einstein, Landau-Lifshitz, Bergmann-Thomson, and Möller prescriptions. The results show that the momentum density components for the Einstein, Bergmann-Thomson, and Möller prescriptions turn out to be same in all cases of the metrics I, II and III, but are different from those of the Landau-Lifshitz prescription, while the energy components remain the same for these three prescriptions only in all possible cases of the metrics I and II. We mention here that the Möller energy-momentum distribution is independent of the coupling constant  $\lambda$ ; that is, these results are valid for any teleparallel models.

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## I. INTRODUCTION

One of the most interesting, but challenging, problems in Einstein's theory of general relativity (GR) is the localization of energy. This problem still needs a definite answer due to its unusual nature and the various viewpoints on it. Among all available theories of gravitation in the literature, GR has been accepted as the true theory of gravitation as many physical aspects of nature have been experimentally verified in the context of this theory. However, the localization of energy and momentum [1] is still an open, unresolved and disputed problem in GR. In GR, many attempts have been made to resolve this problem, but no definition has generally been accepted till now.

As a pioneer, Einstein used the notion of an energy-momentum complex to solve this problem [2]. Following Einstein, many scientists like Landau-Lifshitz [3], Papapetrou [4], Bergmann-Thomson [5], Tolman [6], Weinberg [7] and Möller [8] have introduced their own energy-momentum complexes. All these prescriptions, except Möller's, are restricted to carrying out the calculations in Cartesian coordinates only for the sake of physical results. Also, we cannot define angular momentum with the help of all these prescriptions. Misner et al. [1] showed that energy can only be localized in spherical systems, but very soon after that Cooperstock and Sarracino [9] proved that if energy is localizable for spherical systems, then it can be localized in any coordinate system. Bondi [10] argued that a non-localizable form of energy is not admissible in GR. After this, the idea of quasi-local energy was introduced by Penrose and other scientists [11-13]. In this method, one can use any coordinate system while finding the quasi-local masses to obtain the energy-momentum of a curved spacetime. Bergqvist [14] considered seven different definitions of quasi-local masses and showed that no two of these definitions gave the same results. Chang et al. [15] proved that every energy-momentum complex could be associated with a particular Hamiltonian boundary term. Thus, the energy-momentum complexes may also be considered as quasi-local. Xulu [16-17] extended this investigation and found the same energy distribution in the cases of the Melvin magnetic and the Bianchi type I universes.

Virbhadra and his collaborators [18-21] verified for asymptotically-flat spacetimes that different energy-momentum complexes could give the same result for a given spacetime. They also found encouraging results for the case of asymptotically non-flat spacetimes by using different energy-momentum complexes. Aguirregabiria et al. [22], by using the Ein-

stein, Landau Lifshitz, Papapetrou, Bergmann, and Weinberg (E,LL,P,B,W) prescriptions, showed that the energy distributions within a Kerr-Schild metric were the same. Virbhadra [23] found that these five different prescriptions (E,LL,P,B,W) did not give the same results for the most general non-static spherically-symmetric spacetime. One of the authors found several examples that did not provide the same result for different prescriptions [24]. The results found in [17, 19, 21 and 23-25] lead us to know that the energy distribution in Möller's prescription is different from Einstein's energy for some particular spacetimes, including the Schwarzschild spacetime.

Some authors [26-30] argued that this problem of energy might be settled in the context of the teleparallel theory (TPT) of gravity. They showed that energy-momentum could also be localized in the framework of this theory. The results of the two theories have been shown to agree with each other. Vargas [28] found that the total energy of the closed Friedmann-Robertson-Walker spacetime was zero by using the teleparallel versions of the Einstein and the Landau-Lifshitz complexes. This agrees with the result obtained by Rosen [31] in GR. Salti and his co-workers [29] considered some particular spacetimes and calculated the energy-momentum densities by using different prescriptions both in GR and TPT and they found the similar results.

Sharif and Amir [32-38] evaluated the energy-momentum distribution of the Lewis-Papapetrou spacetime by using the TP version of Möller's prescription and found that the results did not agree with those available in the context of GR. We use the TP versions of the Möller, Einstein, Landau-Lifshitz and Bergmann-Thomson prescriptions to find the energy-momentum distribution of this metric and compare the results with those already found in GR. However, the energy-momentum density components become the same in both theories under certain assumptions. They also discussed the energy-momentum of static, axially-symmetric spacetimes in the framework of teleparallel gravity (TPG). For this purpose, they used the TP versions of the Einstein, Landau-Lifshitz, Bergmann-Thomson and Möller prescriptions. A comparison of the results shows that the energy densities are different, but the momentum turns out to be constant in each prescription. This is exactly similar to the results available in literature when using the framework of GR.

Sharif and his collaborators [39-41] found that the results for the energy exactly coincided with several prescriptions in GR. Interestingly, our results exactly coincide with different energy-momentum prescriptions in GR. The constant momentum shows consistency with the

results available in GR and TPG. Recently, Amir et al. [42] explored the energy-momentum distribution of non-static plane symmetric spacetimes in GR and (TPG).

The scheme of this paper is as follows. In Section **II**, we give some basics of the TPT and the TP versions of the Einstein, Landau-Lifshitz, Bergmann-Thomson and Möller prescriptions. Section **III** is devoted to evaluating of the energy-momentum density components for locally rotationally- symmetric spacetimes. In the last section, we shall summarize the results.

## II. TELEPARALLEL GRAVITY AND ENERGY-MOMENTUM COMPLEXES

First, we briefly outline the main points of the TPT. The basic entity of the TPG is the non-trivial tetrad  $h^a{}_\mu$ , whose inverse is denoted by  $h_a{}^\nu$ , satisfying the relations [34]

$$h^a{}_\mu h_a{}^\nu = \delta_\mu{}^\nu; \quad \text{and} \quad h^a{}_\mu h_b{}^\mu = \delta^a{}_b. \quad (1)$$

The theory of TPG is described by the Weitzenböck connection, which is given as

$$\Gamma^\theta{}_{\mu\nu} = h_a{}^\theta \partial_\nu h^a{}_\mu \quad (2)$$

and is obtained due to the condition of absolute parallelism [35]. This implies that the spacetime structure underlying a translational gauge theory is naturally endowed with a teleparallel structure [35-36]. In this paper, the Latin alphabet ( $a, b, c, \dots = 0, 1, 2, 3$ ) will be used to denote the tangent space indices and the Greek alphabet ( $\mu, \nu, \rho, \dots = 0, 1, 2, 3$ ) to denote the spacetime indices. The Riemannian metric in TPT arises as a byproduct [35] of the tetrad field given by

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu, \quad (3)$$

where  $\eta_{ab}$  is the Minkowski spacetime such that  $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$ .

In TPT, the gravitation is attributed to torsion [36], which plays the role of a force here. For the Weitzenböck spacetime, the torsion is defined as [37]

$$T^\theta{}_{\mu\nu} = \Gamma^\theta{}_{\nu\mu} - \Gamma^\theta{}_{\mu\nu}, \quad (4)$$

which is antisymmetric in nature. Due to the requirement of absolute parallelism, the curvature of the Weitzenböck connection vanishes identically [34]. The Weitzenböck connection

and the Christoffel symbols satisfy the following relation:

$$\Gamma^{0\theta}_{\mu\nu} = \Gamma^\theta_{\mu\nu} - K^\theta_{\mu\nu}, \quad (5)$$

where  $\Gamma^{0\theta}_{\mu\nu}$  are the Christoffel symbols and  $K^\theta_{\mu\nu}$  denotes the **contorsion tensor** and is given by

$$K^\theta_{\mu\nu} = \frac{1}{2}[T^\theta_{\mu\nu} + T^\theta_{\nu\mu} - T^\theta_{\mu\nu}]. \quad (6)$$

The teleparallel versions of the Einstein, Landau-Lifshitz and Bergmann energy-momentum complexes, by setting  $c = 1 = G$ , are, respectively, given by [28]

$$\begin{aligned} hE^\mu_\nu &= \frac{1}{4\pi} \partial_\lambda (U_\nu^{\mu\lambda}), \\ hL^{\mu\nu} &= \frac{1}{4\pi} \partial_\lambda (hg^{\mu\beta} U_\beta^{\nu\lambda}), \\ hB^{\mu\nu} &= \frac{1}{4\pi} \partial_\lambda (g^{\mu\beta} U_\beta^{\nu\lambda}), \end{aligned} \quad (7)$$

where  $U_\nu^{\mu\lambda}$  is Freud's superpotential given as

$$U_\nu^{\mu\lambda} = hS_\nu^{\mu\lambda}. \quad (8)$$

Here,  $S^{\nu\mu\lambda}$  is a tensor quantity that is skew symmetric in its last two indices and is defined as

$$S^{\nu\mu\lambda} = m_1 T^{\nu\mu\lambda} + \frac{m_2}{2} (T^{\mu\nu\lambda} - T^{\lambda\nu\mu}) + \frac{m_3}{2} (g^{\nu\lambda} T^{\beta\mu}_\beta - g^{\mu\nu} T^{\beta\lambda}_\beta), \quad (9)$$

where  $m_1$ ,  $m_2$  and  $m_3$  are three dimensionless coupling constants of TPG [35]. we mentioned here that  $hE^0_0$ ,  $hL^{00}$  and  $hB^{00}$  are the energy densities,  $hE^0_i$ ,  $hL^{0i}$  and  $hB^{0i}$  ( $i = 1, 2, 3$ ) are the momentum densities, and  $hE^i_0$ ,  $hL^{i0}$  and  $hB^{i0}$  are the energy current densities of the Einstein, Landau-Lifshitz and Bergmann prescriptions, respectively.

The Teleparallel equivalent of GR may be obtained by considering the following particular choice [35]:

$$m_1 = \frac{1}{4}, \quad m_2 = \frac{1}{2}, \quad m_3 = -1. \quad (10)$$

The superpotential of Möller's tetrad theory was given by Mikhail et al. [26] as

$$U_\mu^{\nu\beta} = \frac{\sqrt{-g}}{2\kappa} P^{\tau\nu\beta}_{\chi\rho\sigma} [V^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} K^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} K^{\sigma\rho\chi}], \quad (11)$$

where

$$P_{\chi\rho\sigma}^{\tau\nu\beta} = \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\beta} + \delta_{\rho}^{\tau} g_{\sigma\chi}^{\nu\beta} - \delta_{\sigma}^{\tau} g_{\chi\rho}^{\nu\beta}, \quad (12)$$

with  $g_{\rho\sigma}^{\nu\beta}$  being a tensor quantity and being defined as

$$g_{\rho\sigma}^{\nu\beta} = \delta_{\rho}^{\nu} \delta_{\sigma}^{\beta} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\beta}, \quad (13)$$

$K^{\sigma\rho\chi}$  is the contortion tensor as given by Eq. (6),  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$ ,  $\lambda$  is the free dimensionless coupling constant of TPG,  $\kappa$  is the Einstein constant and  $V_{\mu}$  is the basic vector field given by

$$V_{\mu} = T^{\nu}_{\nu\mu}. \quad (14)$$

Now, we can write the Möller energy, momentum, and energy current densities as follows:

$$\Xi_{\mu}^{\nu} = U_{\mu}^{\nu\rho},_{\rho}, \quad (15)$$

where the comma means ordinary differentiation. Further,  $\Xi_0^0$ ,  $\Xi_i^0$  and  $\Xi_0^i$  are the energy, momentum, and energy current densities, respectively, in Möller's prescription.

### III. ENERGY-MOMENTUM DISTRIBUTION OF LRS SPACETIMES

Many authors [27-30] studied extensively the LRS spacetimes that contain well-known exact solutions of the Einstein's field equations. They admit a group of motions  $G_4$  acting multiply transitively on three dimensional non-null orbits, spacelike ( $S_3$ ) or timelike ( $T_3$ ), and the isotropy group is a spatial rotation. These spacetimes are represented by three families of metrics given as [26-27]

$$ds^2 = \epsilon[-dt^2 + A^2(t)dx^2] - B^2(t)dy^2 - B^2(t)\Sigma^2(y, k)dz^2, \quad (16)$$

$$ds^2 = \epsilon[-dt^2 + A^2(t)dx^2] - e^{2x}B^2(t)(dy^2 + dz^2), \quad (17)$$

$$ds^2 = \epsilon[-dt^2 + A^2(t)\{dx - \Lambda(y, k)dz\}^2] - B^2(t)dy^2 - B^2(t)\Sigma^2(y, k)dz^2, \quad (18)$$

where  $k = -1, 0, 1$ ,  $\epsilon = \pm 1$ ,

$$\Sigma = \begin{cases} \sin y, & k = 1, \\ y, & k = 0, \\ \sinh y, & k = -1, \end{cases}$$

and

$$\Lambda = \begin{cases} \cos y, & k = 1, \\ \frac{y^2}{2}, & k = 0, \\ \cosh y, & k = -1. \end{cases}$$

The static and the non-static solutions correspond to  $\epsilon = 1$  and  $\epsilon = -1$ , respectively. We restrict our attention the non-static case as the results for the static case can be obtained consequently. For  $\epsilon = -1$ , the above equations take the forms

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - B^2(t)\Sigma^2(y, k)dz^2, (Metric - I) \quad (19)$$

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{2x}dy^2 - B^2(t)e^{2x}dz^2, (Metric - II) \quad (20)$$

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - \{A^2(t)\Lambda^2(y, k) + B^2(t)\Sigma^2(y, k)\}dz^2 + 2A^2(t)\Lambda(y, k)dx dz. (Metric - III). \quad (21)$$

The metrics in Eq. (7) become Bianchi type  $I(BI)$  or  $VII_0 (BVII_0)$  for  $k = 0$ ,  $III (BIII)$  for  $k = -1$ , and Kantowski-Sachs (KS) for  $k = +1$ . The metrics in Eq. (8) represent Bianchi type  $V(BV)$  or  $VII_h (BVII_h)$  metrics. The metrics in Eq. (9) turn out to be Bianchi types  $II(BII)$  for  $k = 0$ ,  $VIII(BVIII)$  or  $III(BIII)$  for  $k = -1$ , and  $IX(BIX)$  for  $k = +1$ . Now, we will discuss the energy-momentum distribution for the three possible cases arising from the metric in Eq. (19) for different values of  $k$  and from the metric in Eq. (20).

### A. Energy-Momentum Densities of the Metric-I

In this section, we explore the energy-momentum distribution of the metric in Eq. (19) by using the Einstein, Landau-Lifshitz and Bergmann-Thomson and Möller Prescriptions. Three cases are  $k = 0, k = 1$ , and  $k = -1$

**Case I** ( $k = 0$ ): In this case, the tetrad of the metric in Eq. (19) can be written as

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A(t) & 0 & 0 \\ 0 & 0 & B(t) & 0 \\ 0 & 0 & 0 & yB(t) \end{bmatrix}, \quad (22)$$

and its inverse becomes

$$h_a{}^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{A(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{B(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{yB(t)} \end{bmatrix}. \quad (23)$$

Here,

$$h = \det h_\mu^a = \sqrt{-g} = A(t)B^2(t)y.$$

Using Eqs. (22) and (23) in Eq. (2), we get the following non-zero components of the Weitzenböck connections:

$$\Gamma^1{}_{10} = \frac{A'(t)}{A(t)}, \quad \Gamma^2{}_{20} = \frac{B'(t)}{B(t)}, \quad \Gamma^3{}_{30} = \frac{B'(t)}{B(t)}, \quad \Gamma^3{}_{32} = \frac{1}{y}. \quad (24)$$

The corresponding non-vanishing components of the torsion tensor found by using Eq. (24) in Eq. (4) in contravariant form are

$$\begin{aligned} T^{110} &= \frac{A'(t)}{A^3(t)} = -T^{101}, & T^{220} &= \frac{B'(t)}{B^3(t)} = -T^{202}, \\ T^{330} &= \frac{B'(t)}{y^2 B^3(t)} = -T^{303}, & T^{332} &= -\frac{1}{y^3 B^4(t)} = -T^{323}. \end{aligned} \quad (25)$$

Making use of Eq. (25) in Eq. (9) and then multiplying by  $g_{\mu\nu}$ , we find the nonzero components of the  $S$  tensor, in mixed form, to be

$$\begin{aligned} S_0{}^{02} &= -\frac{1}{2yB^2(t)} = S_1{}^{12}, & S_1{}^{01} &= -\frac{B'(t)}{B(t)}, \\ S_2{}^{02} &= -\frac{(A'(t)B(t) + A(t)B'(t))}{2A(t)B(t)} = S_3{}^{03}. \end{aligned} \quad (26)$$

Substituting Eq. (26) in Eq. (8) yields the required non-vanishing components of Freud's superpotential as

$$\begin{aligned} U_0{}^{02} &= -\frac{A(t)}{2} = U_1{}^{12}, & U_1{}^{01} &= -yA(t)B'(t)B(t), \\ U_2{}^{02} &= -\frac{yB(t)(B(t)A'(t) + A(t)B'(t))}{2} = U_3{}^{03}. \end{aligned} \quad (27)$$

If the values from Eq. (27) are substituted in Eq. (7), the non-vanishing energy and momentum density components can be found in TPT by using the Einstein, Landau-Lifshitz



and Bergmann-Thomson prescriptions. For the **Einstein prescription**, the non-zero component of the momentum turns out to be

$$hE_2^0 = \frac{B(t)A'(t) + A(t)B'(t)}{8\pi B(t)}, \quad (28)$$

For the **Landau-Lifshitz prescription**, the existing components of the energy and the momentum are

$$\begin{aligned} hL^{00} &= -\frac{A^2(t)B^2(t)}{8\pi}, \\ hL^{20} &= \frac{yA(t)B(t)(A'(t)B(t) + A(t)B'(t))}{4\pi}. \end{aligned} \quad (29)$$

For the **Bergmann-Thomson prescription**, the surviving momentum component is

$$hB^{20} = \frac{A'(t)B(t) + A(t)B'(t)}{8\pi B(t)}. \quad (30)$$

Now, we explore the energy-momentum distribution by using the TP version of the **Möller Prescription**. For this purpose, we evaluate the non-vanishing components of the contorsion tensor by using Eq. (25) in Eq. (6) in contravariant:

$$\begin{aligned} K^{101} &= \frac{A'(t)}{A^3(t)} = -K^{011}, \quad K^{202} = \frac{B'(t)}{B(t)} = -K^{022}, \\ K^{303} &= \frac{B'(t)}{B(t)} = -K^{033}, \quad K^{323} = -\frac{1}{y^3 B^4(t)} = -K^{233}. \end{aligned} \quad (31)$$

Making use of the Eq. (25) in Eq. (14), we have the non-zero components of the basic vector part as

$$V^0 = -\frac{A'(t)B(t) + 2A(t)B'(t)}{A(t)B(t)}, \quad V^2 = \frac{1}{B^2(t)y}. \quad (32)$$

The required non-vanishing components of the superpotential in Möller's tetrad theory can be easily evaluated from Eq. (11) as

$$\begin{aligned} U_1^{01} &= -\frac{2A(t)B'(t)B(t)y}{\kappa}, \quad U_0^{02} = -\frac{A(t)}{\kappa}, \quad U_1^{21} = \frac{A(t)}{\kappa} = U_0^{20}, \\ U_2^{02} &= -\frac{B(t)y(A'(t)B(t) + A(t)B'(t))}{\kappa} = U_3^{03}, \\ U_3^{23} &= -\frac{\lambda A(t)}{\kappa}. \end{aligned} \quad (33)$$

If we substitute the values from Eq. (33) in Eq. (15) and then take  $c, G = 1$  (gravitational units), the energy and momentum density components turn out to be

$$\Xi_2^0 = \frac{-B(t)(A'(t)B(t) + A(t)B'(t))}{\kappa}. \quad (34)$$

The above results are summarized in the Table 1.

**Table 1.** Energy-momentum density components for different prescriptions

P	Energy Density	Momentum Density
ES	$hE^{00} = 0$	$hE^{20} = \frac{A'(t)B(t)+A(t)B'(t)}{8\pi B(t)}$
LL	$hL^{00} = -\frac{A^2(t)B^2(t)}{8\pi}$	$hL^{20} = \frac{A(t)B(t)[A'(t)B(t)+A(t)B'(t)]y}{4\pi}$
BT	$hB^{00} = 0$	$hB^{20} = \frac{A(t)B'(t)+A'(t)B(t)}{8\pi B(t)}$
MR	$\Xi^{00} = 0$	$\Xi^{20} = \frac{A'(t)B(t)+A(t)B'(t)}{8\pi B(t)}$

**CaseII** ( $k = 1$ ): In this case, we follow the procedure of caseI and obtain the energy-momentum distribution for the four prescriptions, namely, the Einstein, Landau-Lifshitz and Bergmann-Thomson and Möller prescriptions. For the ***Einstein*** prescription, the components of the energy and the momentum are

$$hE_0^0 = \frac{A(t)\sin y}{8\pi}, \quad hE_2^0 = -\frac{B(t)\cos y(B(t)A'(t) + A(t)B'(t))}{8\pi}. \quad (35)$$

For the ***Landau-Lifshitz*** prescription, the components of the energy and the momentum are

$$hL^{00} = -\frac{A^2(t)B^2(t)\cos 2y}{8\pi},$$

$$hL^{20} = \frac{A(t)B(t)(A'(t)B(t) + A(t)B'(t))\sin 2y}{8\pi}. \quad (36)$$

For the ***Bergmann-Thomson*** prescription, the components of the energy and the momentum are

$$hB^{00} = \frac{A(t)\sin y}{8\pi},$$

$$hB^{20} = \frac{\cos y(B(t)A'(t) + A(t)B'(t))}{8\pi B(t)}. \quad (37)$$

For the ***Möller*** prescription, the components of the energy and the momentum are

$$\Xi_0^0 = \frac{A(t)\sin y}{\kappa},$$

$$\Xi_2^0 = -\frac{B(t)\cos y(B(t)A'(t) + A(t)B'(t))}{2}. \quad (38)$$

The above results are summarized in the Table 2.

**Table 2.** Energy-momentum density components for different prescriptions

P	Energy Density	Momentum Density
ES	$hE^{00} = \frac{A(t)\text{sin}y}{8\pi}$	$hE^{20} = \frac{\text{cos}y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$
LL	$hL^{00} = -\frac{A^2(t)B^2(t)\text{cos}2y}{8\pi}$	$hL^{20} = \frac{A(t)B(t)[A'(t)B(t)+A(t)B'(t)]\text{sin}2y}{8\pi}$
BT	$hB^{00} = \frac{A(t)\text{sin}y}{8\pi}$	$hB^{20} = \frac{\text{cos}y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$
MR	$\Xi^{00} = \frac{A(t)\text{sin}y}{8\pi}$	$\Xi^{20} = \frac{\text{cos}y[A'(t)B(t)+A(t)B'(t)]}{8\pi B(t)}$

**CaseIII** ( $k = -1$ ): In this case, we follow the procedure of caseI and obtain the energy-momentum distribution for the four prescriptions, namely, the Einstein, Landau-Lifshitz and Bergmann-Thomson and Möller prescriptions. For the ***Einstein*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hE_0^0 &= -\frac{A(t)\text{sin}hy}{8\pi}, \\ hE_2^0 &= -\frac{B(t)\text{cosh}y(B(t)A'(t) + A(t)B'(t))}{8\pi}. \end{aligned} \quad (39)$$

For the ***Landau-Lifshitz*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hL^{00} &= \frac{A^2(t)B^2(t)(\text{cosh}^2y - \text{sinh}^2y)}{8\pi}, \\ hL^{20} &= \frac{A(t)B(t)(A'(t)B(t) + A(t)B'(t))\text{sinh}2y}{8\pi}. \end{aligned} \quad (40)$$

For the ***Bergmann-Thomson*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hB^{00} &= -\frac{A(t)\text{sin}hy}{8\pi}, \\ hB^{20} &= \frac{(A'(t)B(t) + A(t)B'(t))\text{cosh}y}{8\pi B(t)}. \end{aligned} \quad (41)$$

For the ***Möller*** prescription, the components of the energy and the momentum are

$$\begin{aligned} \Xi_0^0 &= -\frac{B(t)\text{sin}hy}{\kappa}, \\ \Xi_2^0 &= -\frac{B(t)\text{cosh}y(B(t)A'(t) + A(t)B'(t))}{\kappa}. \end{aligned} \quad (42)$$

The above results are summarized in the Table 3.

**Table 3.** Energy-momentum density components for different prescriptions.

<b>P</b>	<b>Energy Density</b>	<b>Momentum Density</b>
ES	$hE^{00} = -\frac{A(t)\sinh y}{8\pi}$	$hE^{20} = \frac{\cosh y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$
LL	$hL^{00} = -\frac{A^2(t)B^2(t)\cosh 2y}{8\pi}$	$hL^{20} = \frac{A(t)B(t)[A'(t)B(t)+A(t)B'(t)]\sinh 2y}{8\pi}$
BT	$hB^{00} = -\frac{A(t)\sinh y}{8\pi}$	$hB^{20} = \frac{[A'(t)B(t)+A(t)B'(t)]\cosh y}{8\pi B(t)}$
MR	$\Xi^{00} = -\frac{A(t)\sinh y}{8\pi}$	$\Xi^{20} = \frac{\cosh y[A'(t)B(t)+A(t)B'(t)]}{8\pi B(t)}$

## B. Energy-Momentum Densities of the Metric-II

The tetrad of the metric in Eq. (20) can be written as

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A(t) & 0 & 0 \\ 0 & 0 & e^x B(t) & 0 \\ 0 & 0 & 0 & e^x B(t) \end{bmatrix}, \quad (43)$$

and its inverse becomes

$$h_a{}^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{A(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{e^x B(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{e^x B(t)} \end{bmatrix}. \quad (44)$$

Here

$$h = \det h^a{}_\mu = \sqrt{-g} = A(t)B^2(t)e^{2x}.$$

Using Eqs. (43) and (44) in Eq. (2), we get the following non-zero components of the Weitzenböck connections

$$\begin{aligned} \Gamma^1{}_{10} &= \frac{A'(t)}{A(t)}, \quad \Gamma^2{}_{20} = \frac{B'(t)}{B(t)}, \quad \Gamma^3{}_{30} = \frac{B'(t)}{B(t)}, \\ \Gamma^3{}_{31} &= 1, \quad \Gamma^2{}_{21} = 1. \end{aligned} \quad (45)$$

The corresponding non-vanishing components of the torsion tensor in contravariant form are

$$\begin{aligned} T^{110} &= \frac{A'(t)}{A^3(t)} = -T^{101}, \quad T^{220} = \frac{B'(t)}{e^{2x}B^3(t)} = -T^{202}, \\ T^{330} &= \frac{B'(t)}{e^{2x}B^3(t)} = -T^{303}, \quad T^{331} = -\frac{1}{e^{2x}A^2(t)B^2(t)} = -T^{313}, \\ T^{221} &= -\frac{1}{e^{2x}A^2(t)B^2(t)} = -T^{212}. \end{aligned} \quad (46)$$

Making use of Eq. (46) in Eq. (9) and then multiplying by  $g_{\mu\nu}$ , we find the non-zero components of the  $S$  tensor, in mixed form, to be

$$\begin{aligned} S_0^{01} &= -\frac{1}{A^2(t)}, \quad S_1^{00} = -\frac{A(t)(A'(t)B(t) + A(t)B'(t))}{2B(t)}, \\ S_1^{01} &= \frac{(A(t)(A'(t) + 2))}{2A^4(t)}. \end{aligned} \quad (47)$$

Substituting Eq. (47) in Eq. (8) yields the required non-vanishing components of Freud's superpotential:

$$\begin{aligned} U_0^{01} &= -\frac{e^{2x}B^2(t)}{A(t)}, \quad U_1^{01} = -e^{2x}A(t)B'(t)B(t), \\ U_2^{02} &= -\frac{e^{2x}B(t)(B(t)A'(t) + A(t)B'(t))}{2} = U_3^{03} \\ U_2^{12} &= \frac{e^{2x}B^2(t)}{2A(t)} = U_3^{13}. \end{aligned} \quad (48)$$

When the values from Eq. (48) are substituted in Eq. (7), the required non-vanishing energy and momentum density components in TPT can be found by using the Einstein's, Landau-Lifshitz and Bergmann-Thomson prescriptions, respectively, as

$$hE_0^0 = -\frac{e^{2x}B^2(t)}{2\pi A(t)}, \quad hE_1^0 = \frac{e^{2x}B(t)B'(t)}{2\pi A(t)}, \quad (49)$$

$$hL^{00} = -\frac{e^{4x}B^4(t)}{\pi}, \quad hL^{10} = \frac{e^{4x}B^3(t)B'(t)}{\pi} \quad (50)$$

and

$$hB^{00} = -\frac{e^{2x}B^2(t)}{2\pi A(t)}, \quad hB^{10} = \frac{e^{2x}B(t)B'(t)}{2\pi A(t)}. \quad (51)$$

Now, we explore the energy-momentum distribution by using the TP version of the **Möller Prescription**. For this purpose, we evaluate the non-vanishing components of the contorsion tensor by using Eq. (46) in Eq. (6) in contravariant form as

$$\begin{aligned} K^{101} &= \frac{A'(t)}{A^3(t)} = -K^{011}, \quad K^{202} = \frac{B'(t)}{e^{2x}B(t)} = -K^{022}, \\ K^{303} &= \frac{B'(t)}{e^{2x}B^3(t)} = -K^{033}, \quad K^{313} = -\frac{1}{e^{2x}A^2(t)B^2(t)} = -K^{133}, \\ K^{122} &= \frac{1}{e^{2x}A^2(t)B^2(t)} = -K^{212}. \end{aligned} \quad (52)$$

The non-vanishing components of the basic vectors are evaluated, and the vector part is

$$V^0 = -\frac{A'(t)B(t) + 2A(t)B'(t)}{A(t)B(t)}, \quad V^1 = \frac{2}{A^2(t)}.$$

The required non-vanishing components of the superpotential, in Möller's tetrad theory, are

$$\begin{aligned}
U_0^{01} &= \frac{-2e^{2x}B^2(t)}{A(t)\kappa}, & U_1^{01} &= -\frac{2A(t)B'(t)B(t)e^{2x}}{\kappa}, \\
U_0^{02} &= -\frac{A(t)}{\kappa}, & U_2^{02} &= -\frac{e^{2x}B(t)(A'(t)B(t) + A(t)B'(t))}{\kappa} = U_3^{03}, \\
U_3^{13} &= \frac{3e^{2x}B^2(t)}{A(t)\kappa} = U_2^{12}.
\end{aligned} \tag{53}$$

Substituting these results in Eq. (15) and using  $c, G = 1$  it yield the energy and the momentum densities in Möller's prescription:

$$\Xi_0^0 = \frac{-4B^2(t)e^{2x}}{A(t)\kappa}, \quad \Xi_1^0 = \frac{-4A(t)B(t)B'(t)e^{2x}}{\kappa}. \tag{54}$$

The above results are summarized in the Table 4.

**Table 4.** Energy-momentum densities components for Different Prescriptions.

P	Energy Density	Momentum Density
ES	$hE^{00} = -\frac{e^{2x}B^2(t)}{2\pi A(t)}$	$hE^{10} = \frac{e^{2x}B(t)B'(t)}{2\pi A(t)}$
LL	$hL^{00} = -\frac{e^{4x}B^4(t)}{\pi}$	$hL^{10} = \frac{e^{4x}B^3(t)B'(t)}{\pi}$
BT	$hB^{00} = -\frac{e^{2x}B^2(t)}{2\pi A(t)}$	$hB^{10} = \frac{e^{2x}B(t)B'(t)}{2\pi A(t)}$
MR	$\Xi^{00} = -\frac{e^{2x}B^2(t)}{A(t)2\pi}$	$\Xi^{10} = \frac{e^{2x}B(t)B'(t)}{2\pi A(t)}$

### C. Energy-Momentum Densities of the Metric-III

In this section, we explore the energy-momentum distribution of the metric in Eq. (21) by using the TP version of the Einstein, Landau-Lifshitz and Bergmann-Thomson prescriptions for the three cases:  $\alpha$ ),  $k = 0$ ,  $\beta$ ),  $k = 1$ , and  $\gamma$ ),  $k = -1$ .

**Case $\alpha$ :** ( $k = 0$ ): In this case, the tetrad of the metric in Eq. (21) can be written as

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A(t) & 0 & \frac{y^2 A(t)}{2} \\ 0 & 0 & B(t) & 0 \\ 0 & 0 & 0 & yB(t) \end{bmatrix}, \tag{55}$$

and its inverse turns out to be

$$h_a{}^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{A(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{B(t)} & 0 \\ 0 & \frac{-y}{2B(t)} & 0 & \frac{1}{yB(t)} \end{bmatrix}. \tag{56}$$

Here,

$$h = \det h^a{}_\mu = \sqrt{-g} = A(t)B^2(t)y.$$

Using Eqs. (55) and (56) in Eq. (2), we get the following non-zero components of the Weitzenböck connections:

$$\begin{aligned}\Gamma^1{}_{10} &= \frac{A'(t)}{A(t)}, & \Gamma^1{}_{30} &= \frac{y^2(A'(t)B(t) - A(t)B'(t))}{2A(t)B(t)}, \\ \Gamma^2{}_{20} &= \frac{B'(t)}{B(t)}, & \Gamma^3{}_{30} &= \frac{B'(t)}{B(t)}, & \Gamma^3{}_{32} &= \frac{1}{y}.\end{aligned}\tag{57}$$

The corresponding non-vanishing components of the torsion tensor can be found, by using Eq. (57) in Eq. (4), which is antisymmetric in nature in contravariant form as

$$\begin{aligned}T^{110} &= \frac{4B^3(t)A'(t) + y^2A^3(t)B'(t)}{4A^3(t)B^3(t)} = -T^{101}, \\ T^{103} &= \frac{B'(t)}{2B^3(t)} = -T^{130}, & T^{220} &= \frac{B'(t)}{B^3(t)} = -T^{202}, \\ T^{330} &= \frac{B'(t)}{y^2B^3(t)} = -T^{303}, & T^{112} &= \frac{y}{4B^4(t)} = -T^{121}, \\ T^{312} &= \frac{1}{2yB^4(t)} = -T^{321}, & T^{123} &= \frac{1}{2yB^4(t)} = -T^{132}, \\ T^{301} &= \frac{B'(t)}{2B^3(t)} = -T^{301}, & T^{332} &= -\frac{1}{y^2B^4(t)} = T^{323}.\end{aligned}\tag{58}$$

Using the same procedure as used for the metric in Eq. (19), we have evaluated the non-vanishing energy-momentum density components for the Einstein Landau-Lifshitz and Bergmann-Thomson prescription (in TPT) as

$$hE_2^0 = -\frac{B(t)(B(t)A'(t) + A(t)B'(t))}{8\pi},\tag{59}$$

$$\begin{aligned}hL^{00} &= -\frac{A^2(t)B^2(t)}{8\pi}, \\ hL^{20} &= \frac{yA(t)B(t)(A'(t)B(t) + A(t)B'(t))}{4\pi}\end{aligned}\tag{60}$$

and

$$hB^{20} = \frac{A'(t)B(t) + A(t)B'(t)}{8\pi B(t)}.\tag{61}$$

Now, we explore the energy-momentum distribution by using the TP version of the **Möller prescription**. For this purpose, we evaluate the non-vanishing components of the contorsion

tensor by using Eq. (58) in Eq. (6) in contravariant form as

$$\begin{aligned}
K^{101} &= \frac{A'(t)}{A^3(t)} + \frac{y^2 B'(t)}{4B^3(t)} = -K^{011}, & K^{202} &= \frac{B'(t)}{B^3(t)} = -K^{022}, \\
K^{301} &= -\frac{B'(t)}{2B^3(t)} = -K^{103}, & K^{303} &= \frac{B'(t)}{y^2 B^3(t)} = -K^{033}, \\
K^{323} &= -\frac{1}{y^2 B^4(t)} = -K^{233}, & K^{211} &= \frac{-y}{4B^4(t)} = -K^{121}, \\
K^{312} &= -\frac{1}{2yB^4(t)} = -K^{132}.
\end{aligned} \tag{62}$$

Making use of Eq. (58) in Eq. (14), we have the non-zero components of the basic vector part:

$$V^0 = -\frac{A'(t)B(t) + 2A(t)B'(t)}{A(t)B(t)}, \quad V^2 = \frac{1}{B^2(t)y}. \tag{63}$$

The required non-vanishing components of the superpotential in Möller's tetrad theory can be easily evaluated from Eq. (11) as

$$\begin{aligned}
U_1^{01} &= -\frac{2A(t)B'(t)B(t)y}{\kappa} = -U_1^{10}, & U_0^{02} &= -\frac{A(t)}{\kappa} = -U_0^{20}, \\
U_3^{01} &= y^2 B(t) \left( \frac{B(t)(A'(t) - A(t)B'(t))}{2\kappa} \right) = -U_3^{10}, \\
U_2^{02} &= -\frac{B(t)y(A'(t)B(t) + A(t)B'(t))}{\kappa} = U_3^{03}, \\
U_3^{23} &= -\frac{\lambda A(t)}{\kappa}, & U_1^{21} &= \frac{A(t)}{\kappa} = U_0^{20}, \\
U_3^{12} &= -y^2 A(t) \left( \frac{y^2(1+\lambda)(A^2(t) + 4\lambda B^2(t))}{8\kappa B^2(t)} \right) = -U_3^{21}, \\
U_1^{21} &= A(t) \left( \frac{y^2(1+\lambda)(A^2(t) + 4\lambda B^2(t))}{4\kappa B^2(t)} \right) = -U_1^{12}, \\
U_2^{13} &= \frac{(-1+\lambda)A(t)}{2\kappa} = -U_2^{31}, & U_1^{23} &= -\frac{(1+\lambda)A^2(t)}{2\kappa B^2(t)} = -U_1^{32}, \\
U_3^{23} &= -\frac{y^2(1+\lambda)A^2(t)}{4\kappa B^2(t)} = -U_3^{32}.
\end{aligned} \tag{64}$$

Substituting the values from Eq. (64) in Eq. (15) and then taking  $c, G = 1$  (gravitational units), we find the energy and momentum density components to be

$$\Xi_2^0 = \frac{-B(t)(A'(t)B(t) + A(t)B'(t))}{\kappa}. \tag{65}$$

The above results are summarized in the Table 5.



**Table 5.** Energy-momentum density components in different prescriptions.

P	Energy Density	Momentum Density
ES	$hE^{00} = 0$	$hE^{20} = \frac{A'(t)B(t)+A(t)B'(t)}{8\pi B(t)}$
LL	$hL^{00} = -\frac{A^2(t)B^2(t)}{8\pi}$	$hL^{20} = \frac{A(t)B(t)[A'(t)B(t)+A(t)B'(t)]y}{4\pi}$
BT	$hB^{00} = 0$	$hB^{20} = \frac{A(t)B'(t)+A'(t)B(t)}{8\pi B(t)}$
MR	$\Xi^{00} = 0$	$\Xi^{20} = \frac{A'(t)B(t)+A(t)B'(t)}{8\pi B(t)}$

**Case $\beta$ :** ( $k = 1$ ): In this case, we follow the same procedure as was used for case I and obtain the energy-momentum distribution for the four prescriptions, namely, the Einstein, Landau-Lifshitz and Bergmann-Thomson prescriptions. For the ***Einstein*** prescription, the components of the energy and the momentum turn out to be

$$hE_2^0 = -\frac{B(t)(B(t)A'(t) + A(t)B'(t))\cos y}{8\pi},$$

$$hE_0^0 = \frac{A(t)\sin y}{8\pi}. \quad (66)$$

For the ***Landau-Lifshitz*** prescription, the components of the energy and the momentum are

$$hL^{00} = 0, \quad (67)$$

$$hL^{20} = \frac{A(t)B(t)\sin y(A'(t)B(t) + A(t)B'(t))\cos y}{4\pi}. \quad (68)$$

For the ***Bergmann-Thomson*** prescription, the components of the energy and the momentum are given as

$$hB^{20} = \frac{(A'(t)B(t) + A(t)B'(t))\cos y}{8\pi B(t)},$$

$$hB^{00} = \frac{A(t)\sin y}{8\pi}. \quad (69)$$

For the ***Möller*** prescription, the non vanishing components of the energy and the momentum are

$$\Xi_2^0 = -\frac{B(t)(B(t)A'(t) + A(t)B'(t))\cos y}{\kappa}. \quad (70)$$

The above results are summarized in the Table 6.

**Table 6.** Energy-momentum density components in different prescriptions

P	Energy Density	Momentum Density
ES	$hE^{00} = \frac{A(t)\sinh y}{8\pi}$	$hE^{20} = \frac{\cosh y[B(t)A'(t) + A(t)B'(t)]}{8\pi B(t)}$
LL	$hL^{00} = 0$	$hL^{20} = \frac{1}{4\pi} \{ A(t)B(t)\sinh y$ $(A'(t)B(t) + A(t)B'(t))\cosh y \}$
BT	$hB^{00} = \frac{A(t)\sinh y}{8\pi}$	$hB^{20} = \frac{\cosh y[B(t)A'(t) + A(t)B'(t)]}{8\pi B(t)}$
MR	$\Xi^{00} = 0$	$\Xi^{20} = \frac{\cosh y[B(t)A'(t) + A(t)B'(t)]}{8\pi B(t)}$

**Case $\gamma$ :** ( $k = -1$ ): In this case, we follow the same procedure as for case I and obtain the energy-momentum distribution for the four prescriptions, namely, the Einstein, Landau-Lifshitz and Bergmann-Thomson as given prescriptions. For the ***Einstein*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hE_2^0 &= -\frac{B(t)(B(t)A'(t) + A(t)B'(t))\cosh y}{8\pi}, \\ hE_0^0 &= -\frac{A(t)\sinh y}{8\pi}. \end{aligned} \quad (71)$$

For the ***Landau-Lifshitz*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hL^{00} &= -\frac{A^2(t)B^2(t)\sinh^2 y}{4\pi}, \\ hL^{20} &= \frac{A(t)B(t)\sinh y(A'(t)B(t) + A(t)B'(t))\cosh y}{4\pi}. \end{aligned} \quad (72)$$

For the ***Bergmann-Thomson*** prescription, the components of the energy and the momentum are

$$\begin{aligned} hB^{20} &= \frac{A'(t)B(t) + A(t)B'(t)\cosh y}{8\pi B(t)}, \\ hB^{00} &= -\frac{A(t)\sinh y}{8\pi}. \end{aligned} \quad (73)$$

For the ***Möller*** prescription, the non-zero components of the energy and the momentum are

$$\Xi_2^0 = -\frac{B(t)(B(t)A'(t) + A(t)B'(t))\cosh y}{\kappa}. \quad (74)$$

The above results are summarized in the Table 7.

**Table 7.** Energy-momentum density components in different prescriptions.

P	Energy Density	Momentum Density
ES	$hE^{00} = -\frac{A(t)\sinh y}{8\pi}$	$hE^{20} = \frac{\cosh y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$
LL	$hL^{00} = -\frac{A^2(t)B^2(t)}{4\pi}\sinh^2 y$	$hL^{20} = \frac{1}{4\pi}\{A(t)B(t)\sinh y$ $(A'(t)B(t) + A(t)B'(t))\cosh y\}$
BT	$hB^{00} = -\frac{A(t)\sinh y}{8\pi}$	$hB^{20} = \frac{\cosh y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$
MR	$\Xi^{00} = 0$	$\Xi^{20} = \frac{\cosh y[B(t)A'(t)+A(t)B'(t)]}{8\pi B(t)}$

#### IV. SUMMARY AND DISCUSSION

The problem of localization of energy has been re-considered in the framework of TPG by many scientists. Some authors [25-28] showed that energy-momentum can also be localized in this theory. Many examples have been explored by different researchers, and are available in literature. Two, main conclusions have been made. Firstly, the results of TPG agree for some prescriptions in some spacetimes while the same prescriptions yield different results for some other spacetimes.

Vargas [28] found that the total energy of the closed Friedmann-Robertson-Walker model was zero by using the TP version of Einstein and Landau-Lifshitz complexes which agreed with the results of GR [31]. Sharif and his collaborators [24], [33-41] used different prescriptions to determine the energy-momentum distributions for various spacetimes and found that the results of TPG and GR were not consistent. Recently, Amir and his collaborators [42] evaluated the energy-momentum distribution of non-static plane-symmetric spacetimes by using different prescriptions in the context of TPG and GR and showed that the results for the Einstein, Landau-Lifshitz, Bergmann-Thomson prescriptions are the same in both the theories but are different from those obtained when using the Möller prescription.

Now, we have extended this work for the whole family of LRS spacetimes. We consider the three metrics representing the LRS spacetimes and explored the energy-momentum distribution by using the TP version of the Einstein, Landau-Lifshitz, Bergmann-Thomson, and Möller prescriptions. Three cases arise for metric I and III (three different values of  $k$ ) while metric II yields only one case. The energy and the momentum density components of the Einstein, Landau-Lifshitz, Bergmann-Thomson and the Möller prescriptions for all seven cases are given in tables 1-7.

We see that energy and the momentum take well-defined and definite forms for each prescription in all seven cases. Tables 1 – 7 show that the momentum density components of the Einstein, Bergmann-Thomson and Möller prescriptions are the same in all seven cases of the metrics I, II and III while the Landau-Lifshitz prescription yields different results. Further, the energy components of all cases of metrics I and II turn out to be same for the Einstein, Bergmann-Thomson and Möller prescriptions but the energy density components of both the Landau-Lifshitz and the Möller prescriptions have been shown to be different for metric-III.

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